Hypohamiltonian Snarks Have a 5-Flow

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Abstract

It is well known that a snark does not admit a 3-edge colouring, neither a 4-flow, nor a Hamiltonian cycle. A snark is *4-edge-(flow)-critical* if the contraction of any of its edges yields a graph that has a 4-flow; it is *2-vertex critical* if the removal of any two adjacent vertices yields a graph that has a 3-edge-colouring; and *hypohamiltonian* if the removal of any of its vertices yields a Hamiltonian graph. In this paper we show that a snark is 4-edge-critical if and only if it is 2-vertex-critical and also that every hypohamiltonian snark admits a 5-flow, thus providing an answer to a question proposed by Cavicchioli et al. in 2003.

Keywords: Nowhere-zero k-flows, Tutte's Flow Conjectures, Snarks, Critical graphs.

1 Introduction

Let $k > 1$ be an integer, let G be a graph, let D be an orientation of G and let φ be a weight function that associates to each edge of G a positive integer in the set $\{1, 2, \ldots, k-1\}$. The pair (D, φ) is a *(nowhere-zero) k-flow* of G

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if every vertex v of G is *balanced*, i. e., the sum of the weights of all edges leaving v equals the sum of the weights of all edges entering v.

Tutte [14] pioneered the study of k-flows. His first fundamental observation was that, for any planar graph, a k-flow can be obtained from a k-face-colouring and vice-versa (see [15] for a proof). Therefore, the concept of a k-flow generalizes that of a k-face-colouring since it does not depend on an embedding of the graph in a particular surface. Then, Tutte proposed three celebrated conjectures regarding k-flows of graphs that generalize to non-planar graphs three famous theorems on face-colourings of planar graphs; such conjectures are known as the 5-, 4- and 3-Flow Conjectures (see [1, Open Problems 95, 96, 97]). Several partial results regarding Tutte's Conjectures have been established in the past decades; detailed accounts can be found in [1,6,16,11].

In this paper, we will focus on Tutte's 5-Flow Conjecture, which states that every 2-edge-connected graph admits a 5-flow. The most important partial results related to this conjecture are Seymour's 6-Flow Theorem [10] and the Cubic 4-Flow Theorem, due to Robertson, Seymour, Sanders and Thomas [2]. Seymour's 6-Flow Theorem [10] states that every 2-edge-connected graph has a 6-flow. The Cubic 4-Flow Theorem states that every 2-edge-connected cubic graph without a Petersen minor admits a 3-edge-colouring.

It is well known that there is a reduction of Tutte's 5-Flow Conjecture to cubic graphs (see [5]). A famous theorem of Tutte, stated below, is fundamental for the research presented in this paper (see [1, Theorem 21.11] for a proof).

Theorem 1.1 (Tutte) *A cubic graph admits a 4-flow if and only if it admits a 3-edge-colouring.*

As a 4-flow is also a 5-flow, it follows from Theorem 1.1 that no 3-edge-colourable cubic graph is a counterexample to Tutte's 5-Flow Conjecture. A graph is *cyclically 4-edge-connected* if it is necessary to remove at least four edges to obtain at least two connected components with cycles. A cubic non-3-edge-colourable graph that is also cyclically 4-edge-connected and has girth at least five is a *snark*. It is also known that any smallest counterexample to Tutte's 5-Flow Conjecture must be a snark (see [7, Theorem 9.3]). Therefore, proving the 5-Flow Conjecture is reduced to proving that every snark admits a 5-flow.

The study of critical graphs, i. e., graphs that do not have a certain prop-

[2] The proof consists of five papers by various subsets of these authors, of which only paper [9] has been published so far.

erty but that after some reduction operation result in a graph with that property is natural in the quest for mathematical structures that characterize graphs with that property. In this paper, we show how some classes of critical snarks are related.

2 Flow-Critical Graphs

A graph G is *k-(flow)-edge-critical* if it does not admit a k-flow but the graph G/e obtained by the contraction of an arbitrary edge e admits a *k-flow*. The concept of *k-edge-critical* graphs was introduced in [3] with the purpose of capturing the notion of what is a minimal graph without a k-flow and also understanding the structure of such graphs. Clearly, as contracting an edge does not create any new cuts, nor new minors, every minimum counterexample to any of Tutte's Flow Conjectures is a k-edge-critical graph.

It can be shown that a 2-edge-connected graph G without a k-flow is k-edge-critical if and only if, for every edge e of G, the graph $G - e$ admits a k-flow (see [3, Theorem 3.1]). When G is a cubic graph, the graph $G - e$ is a subdivision of another cubic graph, denoted by G_e and called the *underlying cubic graph* of $G - e$. It is well known that a graph G admits a k-flow if and only if any of its subdivisions admit a k-flow. Therefore, Corollary 2.1 stated below follows from these observations and Theorem 1.1.

Corollary 2.1 *Let G be a snark and let e be an edge of G. Then, G is 4-edge-critical if and only if the underlying cubic graph G_e of $G-e$ is 3-edge-colourable for every edge e of G.*

Jaeger [7, Theorem 8.2] observed that a cubic 3-edge-connected graph that has an edge whose removal results in a graph that has a 4-flow must have a 5-flow. In Theorem 2.2 stated below we show that this result can be extended to general k for 2-edge-connected graphs that are not necessarily cubic. It follows as a corollary that every k-edge-critical graph has a $(k+1)$-flow.

Theorem 2.2 *Let G be a 2-edge-connected graph and let e be an edge of G. If $G - e$ admits a k-flow, graph G admits a $(k+1)$-flow.*

Proof Let u and v be the ends of e and let (D', φ') be a k-flow of $G - e$. Let U denote the set of vertices of G that can be reached by directed paths in D' starting at u. By definition, the edge-cut associated with U, denoted $\partial(U)$, cannot have any directed edge leaving U. However, as every vertex in U is balanced with respect to (D', φ'), cut $\partial(U)$ must also be balanced. We conclude that $\partial(U)$ must be empty, and therefore $v \in U$. Let P be a directed path in D' starting at u and ending at v. In the extension D of D' where edge e is directed from v to u, there is a directed cycle given by $P+e$. Let φ be the extension of φ' to G obtained from φ' by adding one unit of flow exclusively to the edges of cycle $P+e$. Clearly, (D, φ) is a $(k+1)$-flow of G. □

Corollary 2.3 *Every 2-edge-connected k-edge-critical graph admits a $(k+1)$-flow.*

It is known that every Hamiltonian cubic graph admits a 3-edge-colouring; therefore, no snark is Hamiltonian. A snark G is *almost Hamiltonian* if there is a vertex v such that $G-v$ is Hamiltonian and *hypohamiltonian* if, for every vertex v, $G-v$ is Hamiltonian.

Corollary 2.4 *Every almost Hamiltonian snark G has a 5-flow.*

Proof By definition, $G-v$ is Hamiltoninan for some vertex v. Then, the underlying cubic graph G_e of any edge e incident with v is also Hamiltonian and has a 4-flow. By Jaeger's Theorem [7, Theorem 8.2] (or by Theorem 2.2) graph G has a 5-flow. □

3 Colour-Critical Snarks

In the past decades researchers have explored several different notions of criticality for snarks, with respect to 3-edge-colourability. We highlight here three such notions defined in [8]. A snark G is *2-vertex-critical* if the removal of an arbitrary pair of adjacent vertices yields a 3-edge-colourable graph, and it is *2-vertex-cocritical* if the removal of an arbitrary pair of non-adjacent vertices yields a 3-edge-colourable graph. A snark is *bicritical* if it is simultaneously 2-vertex-critical and 2-vertex-cocritical.

In 2003, Cavicchioli et al. [2] used a computer program to find out how many of the snarks of order at most 28 were in each of these classes of critical snarks. In 2013, Silva, Pesci and Lucchesi [4] used a computer program to find out how many of the snarks of order at most 28 were 4-edge-critical. The striking equality observed in the numbers of 2-vertex-critical and 4-edge-critical snarks presented in [2, Table II] and [4, Table 1] inspired us to further investigate the existance of an equivalence between such classes. Such equivalence was indeed established, as shown on Theorem 3.1 below.

Theorem 3.1 *A snark is 4-edge-critical if and only if it is 2-vertex-critical.*

Proof Let G be a snark, and let $e = uv$ be an arbitrary edge of G. Assume that the neighbourhoods of u and v are $N(u) = \{v, u_1, u_2\}$ and $N(v) =$

$\{u, v_1, v_2\}$, respectively. Then, the underlying cubic graph G_e of $G - e$ can be obtained from graph $G \setminus \{u, v\}$ by the addition of edges u_1u_2 and v_1v_2.

If snark G is 4-edge-critical, then, by Corollary 2.1, the underlying cubic graph G_e of $G - e$ has a 3-edge-colouring. Then, so does graph $G \setminus \{u, v\}$, a subgraph of G_e. This argument is valid for every pair of adjacent vertices u and v, whence G is 2-vertex-critical.

Assume now that snark G is 2-vertex-critical. All vertices of $G \setminus \{u, v\}$ have degree three, except $\{u_1, u_2, v_1, v_2\}$, which have degree two. By definition, graph $G \setminus \{u, v\}$ has a 3-edge-colouring. Clearly, the set of vertices that miss any given colour is an even subset of $\{u_1, u_2, v_1, v_2\}$.

We assert that vertices u_1 and u_2 miss a common colour. For this, assume the contrary. Without loss of generality, we may assume that u_1 and v_1 miss a common colour, c_1, and u_2 and v_2 a common colour c_2, where $c_1 \neq c_2$. In that case, we may extend the 3-edge-colouring of $G \setminus \{u, v\}$ to G, by assigning to u_iu and v_iv the colour c_i ($i = 1, 2$), and the third colour to edge e, a contradiction.

As vertices u_1 and u_2 miss a common colour, c_u, then vertices v_1 and v_2 also miss a common colour, c_v, possibly equal to c_u. In that case, we extend the 3-edge-colouring of $G \setminus \{u, v\}$ to a 3-edge-colouring of G_e by assigning to u_1u_2 the colour c_u and to v_1v_2 the colour c_v. We therefore conclude that it is always possible to obtain a 3-edge-colouring of G_e, for every edge e of G. By Corollary 2.1, G is 4-edge-critical. □

Steffen [12] showed that every hypohamiltonian snark G is bicritical (therefore 2-vertex-critical). So, it also follows from Theorem 3.1 and Corollary 2.3 that every hypohamiltonian snark has a 5-flow.

4 Concluding Remarks

We showed that a snark is 4-edge-critical if and only if it is 2-vertex-critical. We also showed that every hypohamiltionian snark has a 5-flow, thus providing an answer to a question proposed by Cavicchioli et al. in 2003. Given these relations observed, it is natural to also investigate if other pairs of the classes of critical snarks studied can also be related. We observed that no other relation can be established. According to [2, Table I], there are 2-vertex-critical snarks that are not hypohamiltonian. Also, by [2, Table II], there exist 2-vertex-cocritical snarks that are not 2-vertex-critical. In 1996, Nedela and Skoviera [8] had already raised the question whether the classes of 2-vertex-critical and bicritical snarks were equivalent. However, Steffen [13] showed that there are infinitely many 2-vertex-critical snarks that are not bicritrical.

References

[1] Bondy, J. A. and U. S. R. Murty, "Graph Theory," Springer, 2008.

[2] Cavicchioli, A., T. E. Murgolo, B. Ruini and F. Spaggiari, *Special classes of snarks*, Acta Applicandae Mathematica **76** (2003), pp. 57–88.

[3] da Silva, C. N. and C. L. Lucchesi, *Flow-critical graphs*, Electronic Notes in Discrete Mathematics **30** (2008), pp. 165–170.

[4] da Silva, C. N., L. Pesci and C. L. Lucchesi, *Snarks and flow-critical graphs*, Electronic Notes in Discrete Mathematics **44** (2013), pp. 299–305.

[5] de Almeida e Silva, L. M., "Fluxos Inteiros em Grafos," Master's thesis, Dept. of Computer Science – University of Campinas – Brazil(1991), In Portuguese.

[6] Diestel, R., "Graph Theory," Graduate Texts in Mathematics **173**, Springer-Verlag, 1996.

[7] Jaeger, F., *Nowhere-zero flow problems*, Selected Topics in Graph Theory **3**, Academic Press, 1988 pp. 71–95.

[8] Nedela, R. and M. Škoviera, *Decompositions and reductions of snarks*, Journal of Graph Theory **22** (1996), pp. 253–279.

[9] Robertson, N., P. D. Seymour and R. Thomas, *Tutte's edge colouring conjecture*, J. Comb. Theory, Series B **70** (1997), pp. 166–183.

[10] Seymour, P. D., *Nowhere-zero 6-flows*, J. Comb. Theory, Series B **30** (1981), pp. 130–135.

[11] Seymour, P. D., *Nowhere-zero flows*, in: R. L. Graham, M. Grötschel and L. Lovász, editors, *Handbook of Combinatorics*, Elsevier, 1995 pp. 289–299.

[12] Steffen, E., *Classifications and characterizations of snarks*, Discrete Math. **188** (1998), pp. 183–203.

[13] Steffen, E., *Non-bicritical critical snarks*, Graphs and Combinatorics **15** (1999), pp. 473–480.

[14] Tutte, W. T., *A contribution to the theory of chromatic polynomials*, Can. J. Math. **6** (1954), pp. 80–91.

[15] Younger, D. H., *Integer flows*, Journal of Graph Theory **7** (1983), pp. 349–357.

[16] Zhang, C.-Q., "Integer Flows and Cycle Covers of Graphs," Marcel Dekker, 1997.